Bäcklund transformations, D-branes and fluxes in minimal type 0 strings

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Abstract

We study the type 0A string theory in the $(2, 4k)$ superconformal minimal model backgrounds, focusing on the fully non-perturbative string equations which define the partition function of the model. The equations admit a parameter, Γ, which in the spacetime interpretation controls the number of background D-branes, or R–R flux units, depending upon which weak coupling regime is taken. We study the properties of the string equations (often focusing on the $(2, 4)$ model in particular) and their physical solutions. The solutions are the potential for an associated Schrödinger problem whose wavefunction is that of an extended D-brane probe. We perform a numerical study of the spectrum of this system for varying Γ and establish that when Γ is a positive integer the equations' solutions have special properties consistent with the spacetime interpretation. We also show that a natural solution-generating transformation (that changes Γ by an integer) is the Bäcklund transformation of the KdV hierarchy specialized to (scale invariant) solitons at the zero velocity. Our results suggest that the localized D-branes of the minimal string theories are directly related to the solitons of the KdV hierarchy.

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(Some figures in this article are in colour only in the electronic version)

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1. Background

The $(2, 4k)$ series of the minimal type 0A string theory, in the presence of background R–R sources, has a non-perturbative definition via the following 'string equation' [1]:

$$u\mathcal{R}^2 - \tfrac{1}{2}\mathcal{R}\mathcal{R}'' + \tfrac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2. \tag{1}$$

Here $u(z)$ is a real function of the real variable z; a prime denotes $\nu\partial/\partial z$; and Γ and ν are constants. The quantity $\mathcal{R}$ is defined by

$$\mathcal{R} = \sum_{k=0}^{\infty}\left(k + \frac{1}{2}\right)t_k R_k, \tag{2}$$

where the R_k ($k = 0, \ldots$) are polynomials in $u(z)$ and its z-derivatives. They are related by a recursion relation,

$$R'_{k+1} = \tfrac{1}{4}R'''_k - uR'_k - \tfrac{1}{2}u'R_k, \tag{3}$$

and are fixed by the constant R_0, and the requirement that the rest vanish for vanishing u. The first few are

$$R_0 = \tfrac{1}{2}; \qquad R_1 = -\tfrac{1}{4}u; \qquad R_2 = \tfrac{1}{16}(3u^2 - u''). \tag{4}$$

The kth model is chosen by setting all the other ts to zero except $t_0 \equiv z$, and t_k, the latter being fixed to a numerical value such that $\mathcal{R} = \mathcal{D}_k - z$. The $\mathcal{D}_k$ are normalized such that the coefficient of u^k is unity, e.g.

$$\mathcal{D}_1 = u, \qquad \mathcal{D}_2 = u^2 - \tfrac{1}{3}u'', \qquad \mathcal{D}_3 = u^3 - uu'' - \tfrac{1}{2}(u')^2 + \tfrac{1}{10}u''''. \tag{5}$$

For the kth model, equation (1) has asymptotics

$$\begin{aligned}
u(z) &= z^{\frac{1}{k}} + \frac{\nu\Gamma}{kz^{1-\frac{1}{2k}}} + \cdots && \text{for} \quad z \longrightarrow +\infty, \\
u(z) &= \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} + \cdots && \text{for} \quad z \longrightarrow -\infty.
\end{aligned} \tag{6}$$

The function $u(z)$ defines the partition function $Z = \exp(-F)$ of the string theory via

$$u(z) = \nu^2\frac{\partial^2 F}{\partial\mu^2}\bigg|_{\mu=z}, \tag{7}$$

where μ is the coefficient of the lowest dimension operator in the worldsheet theory. Integrating twice, the asymptotic expansions in equations (6) furnish the partition function perturbatively as an expansion in the dimensionless string coupling

$$g_s = \frac{\nu}{\mu^{1+\frac{1}{2k}}}. \tag{8}$$

The role of Γ is now clear: at the large positive z we find that Γ controls the number of background D-branes since the perturbative series contains both open and closed string worldsheets with a factor of Γ for each boundary; while the large negative z series gives only closed string worldsheets, with Γ^2 appearing when there is an insertion of pure R–R flux [1, 2].

From the point of view of the kth theory, the other t_ks represent coupling to closed string operators $\mathcal{O}_k$. It is well known that the insertion of each operator can be expressed in terms of the KdV flows [3, 4]:

$$\frac{\partial u}{\partial t_k} = R'_{k+1}. \tag{9}$$

The operator $\mathcal{O}_0$ couples to t_0, which is in fact $-4z$, the cosmological constant (in the unitary model). So $\mathcal{O}_0$ is often referred to as the puncture operator, which yields the area of a surface by fixing a point which is then integrated over in the path integral. So $u(z)$ is the two-point function of the puncture operator.

For $\Gamma = 0$ the string equation (1) was discovered by defining a family of string theories using double scaled models of a complex matrix M:

$$\mathcal{Z} = \int \mathrm{d}M \exp\left\{ -\frac{N}{\gamma} \mathrm{Tr}(V(MM^\dagger)) \right\}. \tag{10}$$

The resulting physics captured by the string equation was the first complete non-perturbative definition of a string theory [5–9], sharing the large z perturbation theory of the original (bosonic) string theories obtained by double scaling of Hermitian matrix models [10–12], but not suffering from their non-perturbative shortcomings.

Non-zero Γ can also be studied using a matrix model definition in a variety of ways. One way is to add a logarithmic term [13, 14] to an appropriate [1, 15] matrix model potential. Expanding the logarithm clearly adds holes of all sizes to the string worldsheets defined by studying the duals of the Feynman diagrams of the model. Another (equivalent) method is to define M as an $(N + \Gamma) \times N$ rectangular matrix [2, 16]. The double scaling limit (in which N is taken large) then yields equation (1).

It is clear from these two methods that, in the interpretation of the matrix model as the world-volume theory of N D-branes (which defines the closed string theory holographically [17] after taking the double scaling limit), the modification corresponds to adding Γ 'quark flavours' to the world-volume model. This corresponds to adding sectors of open strings stretching between the N D-branes and Γ extra D-branes.

So from this point of view, it appears that Γ is a positive integer. However, this is not at all clear from the string equation itself. In fact, it is known perturbatively that the solutions of the equation have special properties for various fractional values of Γ.[4] These might well turn out to be unphysical values, but this is not *à priori* clear. One of the purposes of this paper is to show that Γ being a positive integer is in fact selected out as special from the point of view of the string equation. We will do this by studying the equation non-perturbatively using numerical methods, and also by observing the properties of a natural transformation between solutions of the string equation which change Γ by ± 1. This transformation was discovered in [1], and we derive an explicit form for it here, and employ this form in our investigations. We further observe that the transformation is in fact a specialization (to the scale invariant soliton sector) of the Bäcklund transformations of the KdV hierarchy!

These Bäcklund transformations are known to change the soliton number of solutions of the KdV equation (and its higher order analogues) [18], and so our observation is particularly intriguing, since it implies that the (localized [19]) D-branes of the minimal type 0A string theory *are intimately connected with the solitons of the KdV hierarchy*.

2. A spectral problem

A solution $u(z)$ of the string equation serves as the potential for a one-dimensional Hamiltonian $\mathcal{H}$ which arises naturally in the double-scaled matrix model [21]:

$$\mathcal{H} = -Q = -v^2 \frac{\partial^2}{\partial z^2} + u(z), \tag{11}$$

where $\sqrt{2}v$ plays the role of $\hbar$.

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[4] Note, in particular, the case $\Gamma = \pm 1/2$ in the large negative z regime of equation (6). There are several other interesting cases of this type, including a family of exact double pole solutions, for half-integer Γ. See [1, 15].

In fact, the (first derivative of) the string equation can be obtained by eliminating the wavefunction $\psi(z)$ from the equations:

$$Q\psi = \lambda_s \psi, \qquad \widetilde{P}\psi = \lambda_s \frac{\partial}{\partial \lambda_s}\psi. \tag{12}$$

Here, $\widetilde{P}$ is a differential operator representing scale transformations. It is constructed from (the differential operator part of) fractional powers [22, 23] of Q, as follows [6]:

$$\widetilde{P} = \sum_{k=1}^{\infty} \left(k + \frac{1}{2}\right) t_k Q_+^{k+\frac{1}{2}} - \frac{1}{2} z \frac{\partial}{\partial z} + \text{const.} \tag{13}$$

Eliminating ψ, one has the operator equation [6]:

$$[\widetilde{P}, Q] = Q, \tag{14}$$

which yields an equation for $u(z)$ which is the first derivative of equation (1).

The spectrum of this model has interesting and important information, and is a rather direct way of getting access to the non-perturbative physics of the model[5]. The problem is naturally defined in terms of the eigenvalues, λ, of the combination $MM^{\dagger}$. These are naturally positive. So in the double scaling limit (which focuses on the infinitesimal region near the tail of the eigenvalue distribution of the model) the scaled eigenvalues, which we denote as λ_s, are also distributed on the positive real line.

We verified this in the full non-perturbative regime by studying the spectrum of $\mathcal{H}$ numerically. To do this, we solved the differential equation for a given value of k as a boundary value problem, with the perturbative boundary conditions given in equations (6). We started out by using the equation-solving routine dsolve in the package *Maple 9*, although later we used other methods (see below for the case of negative Γ). We used a discrete z lattice of up to 8192 points for the positive Γ.

First, note the form of the function $u(z)$ for some *strictly positive* values of Γ (see figure 1). Since the potential tends to zero at $z \to -\infty$, and rises monotonically in the $z \to +\infty$ limit, it is clear that the spectrum has the chance to be both continuous and bounded from below by zero, as we expect from the perturbation theory. A striking feature is the fact that the potential develops a localized well in the interior, as $\Gamma \to 0$.

It is therefore interesting and important to *non-perturbatively* establish these properties (continuity, boundedness) of the spectrum for various Γ. First, we checked the spectrum of the case $\Gamma = 0$ (the case with no background branes), and verified that there are no bound states, despite the appearance of the well[6,7]. We set up the problem numerically by discretizing the spatial coordinate z, typically using the interval $-10 \leqslant z \leqslant +10$, with $\nu = 1$. We used a range of lattice spacings in studying our problem to ensure stability. This was equivalent to a number of lattice points ranging from as little as 500 up to 10 000 (the latter used when more accuracy was needed, as we shall see later on.) Discretizing this problem turns it into a problem of diagonalizing a tri-diagonal matrix for which we coded a fairly efficient well-known method (the TQLI method [26]) in C++ to do the work. This allowed us to establish non-perturbatively with some confidence that there are no bound states in the well for the

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[5] The wavefunction ψ is a natural extended D-brane [24] probe of the model, a fact which has been studied extensively recently for the bosonic string in [25].

[6] That this is the case exactly fits with a suggestive (but rough) estimate one can make by comparing the depth of the well with the value of the lowest energy eigenstate of the equivalent harmonic oscillator, determined by reading off the value of the second derivative at the bottom of the well (computed numerically). The well falls short in depth by $\sim 10\%$.

[7] This complements earlier studies of the properties of the $\Gamma = 0$ solutions carried out extensively in [5–9].
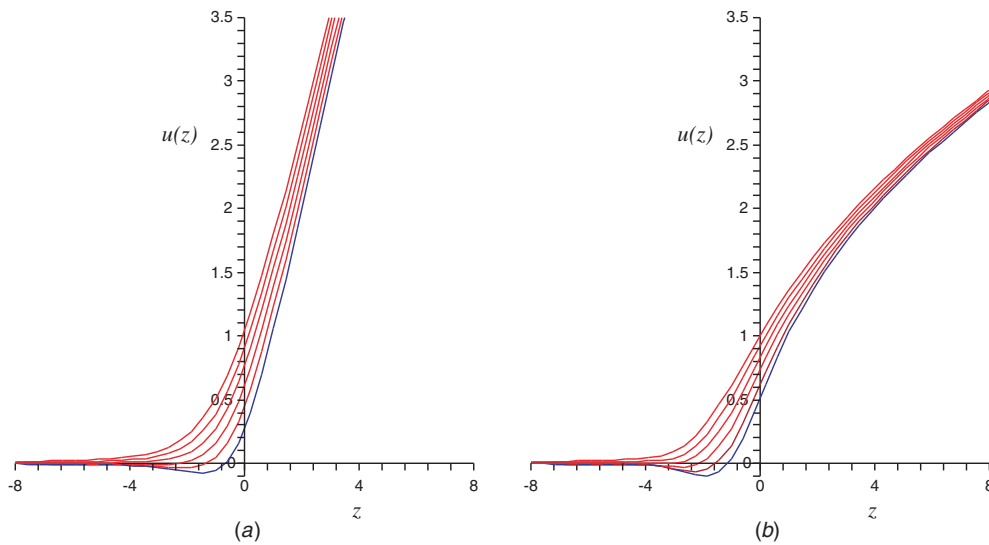
Figure 1. Numerical solutions to equation (1) for $u(z)$: (*a*) the case $k = 1$ and (*b*) the case $k = 2$, for a range of values of the parameter Γ. The very bottom curve in each case has $\Gamma = 0$. The solutions all asymptote to $u = z^{1/k}$ for the large positive z and $u = 0$ for the large negative z.

positive Γ, and so the spectrum is continuous and bounded from below by zero, in accord with perturbative expectations.

The next step is the case of negative Γ, which is clearly allowed by the equation and so should be studied. Our first observation is that solving the differential equation numerically became much more prone to error, and we had to use much more accurate methods. We used the NAG library of routines for solving boundary value problems with C++ to make further progress, as they gave much greater control over the problem. We were able to construct directly in this way a solution for $u(z)$ for values as low as $\Gamma = -0.970$ before running into numerical problems. See figure 2.

Once again we should check the perturbative expectation that the spectrum is positive persists non-perturbatively. Several solutions $u(z)$, for various Γ down to -0.970 were used as potentials for $\mathcal{H}$ and found (upon placing in our tridiagonal matrix diagonalization routine) to have no bound states.

With these methods, the growing instability is suggestive, but not compellingly so, that something interesting might be happening at $\Gamma = -1$. In order to establish this more satisfyingly we employed some exact methods (in the form of a solution-generating transformation), which we will describe next. This will allow us to generate new $u(z)$ from those we already can extract numerically, and allow us to approach $\Gamma = -1$ in a controlled manner, and with a convincing accuracy.

3. Bäcklund transformations

In [1], it was established that the double scaled unitary matrix models of [27, 28] had an interpretation as continuum string theories with open string sectors. This was done by establishing a direct connection between the solutions of the string equations in equation (1)
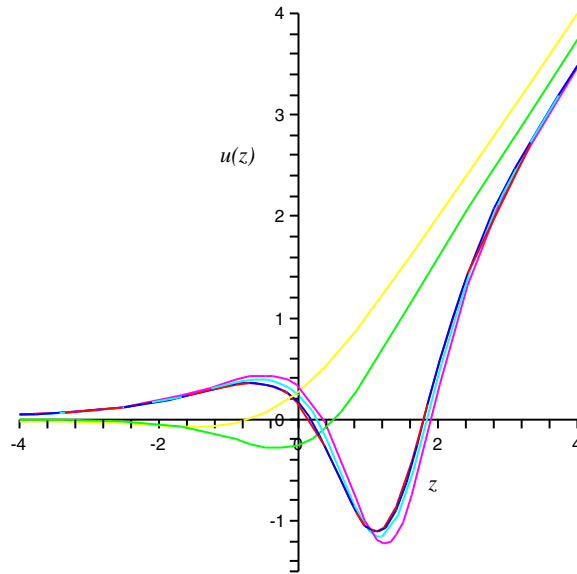
Figure 2. Numerical solutions to equation (1) for $u(z)$ in the case $k = 1$, for a range of negative values of the parameter Γ: $0.000, -0.500, -0.950, -0.965, -0.968$ and -0.970.

and solutions of the string equations for those unitary matrix models, which are most naturally written in terms of the quantities S_k, where

$$S_k \equiv \tfrac{1}{2} R'_k[v^2 + v'] - v R_k[v^2 + v'], \tag{15}$$

as

$$\sum_{k=1}^{\infty} \left(k + \frac{1}{2} \right) t_k \, S_k[v(z)] + z v(z) = vC. \tag{16}$$

The models are organized by the mKdV hierarchy:

$$\frac{\partial v}{\partial t_k} = \frac{1}{2} S'_k[v]. \tag{17}$$

We define $u(z)$ and $v(z)$ such that

$$X_{\pm}[u, v] \equiv \tfrac{1}{2} \mathcal{R}'[u] \mp v\Gamma - v(z)\mathcal{R}[u] = 0, \tag{18}$$

which implies a specific form for $v(z)$ given a $u(z)$

$$v = \frac{\tfrac{1}{2}\mathcal{R}'[u] \mp v\Gamma}{\mathcal{R}[u]}. \tag{19}$$

Noting the identity [1]

$$0 = X_{\pm}^2 \pm v\Gamma X_{\pm} - \mathcal{R}[u]X'_{\pm} \equiv (v^2 + v')\,\mathcal{R}^2[u] - \tfrac{1}{2}\mathcal{R}[u]\,\mathcal{R}''[u] + \tfrac{1}{4}(\mathcal{R}'[u])^2 - v^2\,\Gamma^2, \tag{20}$$

we see that if the inverse transformation $u = v^2 + v'$ (known in the integrable literature as the Miura map) exists, then this is just our original string equation (1); and on substitution into equation (19) gives equation (16) with $C = 1/2 \pm \Gamma$. Since the unitary matrix models of [27, 28] were originally derived with $C = 0$, those models turn out to be identified with the case $\Gamma = 1/2$.

Since the transformation is true for both values of C, we find that a given $u(z)$ for a value of Γ gives two functions $v_C(z)$ and $v_{1-C}(z)$, which must be related by

$$v_C^2 + v_C' = v_{1-C}^2 + v_{1-C}', \tag{21}$$

and hence

$$v_{1-C} = v_C + \frac{2C - 1}{\mathcal{R}\left[v_C^2 + v_C'\right]}. \tag{22}$$

On the other hand, the string equation (16) has the symmetry $v_{-C} = -v_C$, and so we obtain a result which relates $v(z)$ at one value of C to another for $C \pm 1$:

$$v_{C\pm 1} = -v_C - \frac{2C \pm 1}{\mathcal{R}\left[v_C^2 \mp v_C'\right]}. \tag{23}$$

This implies a similar transformation for functions $u(z)$, relating Γ and $\Gamma \pm 1$. We can make this explicit by combining equations (19), (20), (23) and $u = v^2 + v'$ to give the remarkable result:

$$u_{\Gamma\pm 1} = \frac{3(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\,\Gamma\,\mathcal{R}' + 4\nu^2\,\Gamma^2}{4\mathcal{R}^2}, \tag{24}$$

where $\mathcal{R} \equiv \mathcal{R}(u_\Gamma)$.

In the theory of nonlinear differential equations (such as those encountered in the integrable model context) Bäcklund transformations are extremely useful, since they convert a given solution of the differential equation into a new solution[8]. Strictly speaking, it looks rather like this is *not* what we have here since our string equation (1) is displayed with Γ explicitly appearing, so u_Γ and $u_{\Gamma\pm 1}$ are solutions of *different* equations. However, the once-differentiated string equation (which is in fact the one which appears naturally in many derivations; see the beginning of the previous section, and see below), which is

$$\tfrac{1}{2}\mathcal{R}''' - 2u\,\mathcal{R}' - u'\,\mathcal{R} = 0, \tag{25}$$

does not have an explicit appearance of Γ. The solutions u_Γ do know of Γ, of course, since it appears (for example) in their asymptotic expansion (6).

So we have a genuine Bäcklund transformation for our system. What is particularly interesting is that it is not just any such transformation but a special case of the well-known Bäcklund transformations known to change the number of solitons in the KdV hierarchy. We can make this explicit as follows:

We start with the KdV equations (9), which we rewrite as

$$\frac{\partial \tilde{u}}{\partial t_k} = R_{k+1}'[\tilde{u}] = \left(\frac{1}{4}\,\mathrm{d}^3 - \tilde{u}\,\mathrm{d} - \frac{1}{2}\frac{\partial \tilde{u}}{\partial x}\right) R_k[\tilde{u}], \tag{26}$$

where now $\mathrm{d} \equiv \frac{\partial}{\partial x}$.

We can search for solutions of the form[9] $\tilde{u}(x, t_k) = t_k^{2\beta_k} u\left(x t_k^{\beta_k}\right)$. We then find $\beta_k = -1/(2k + 1)$ and that (under appropriate rescaling of t_k) equation (26) becomes

$$-u - \tfrac{1}{2}z\,u' = \left(\tfrac{1}{4}\,\mathrm{d}^3 - u\,\mathrm{d} - \tfrac{1}{2}u'\right) R_k[u], \tag{27}$$

where a prime denotes differentiation with respect to $z = x t^{\beta_k}$. Re-arranging we find the once-differentiated string equation (25) for a particular k.

This is another way of making the observation (already noted in [6]) that the string equation follows a restriction of the KdV hierarchy to scale invariant solutions. This also fits

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[8] These are actually 'auto-Bäcklund' transformations.

[9] More generally we search for solutions of the form $\tilde{u}(x, t_k) = t_k^{\alpha_k} u(x t_k^{\beta_k})$; but we find that consistency forces us to choose $\alpha_k = 2\beta_k$.

with recovery [6] of the string equation from the operator relation $[\widetilde{P}, Q] = Q$, where $\widetilde{P}$ is the generator of scale transformations[10]. (See the previous section.)

Now note that the KdV hierarchy admits the following well-known Bäcklund transformation relating a solution $\tilde{u}_1$ to a solution $\tilde{u}_2$:

$$w_x + y_x = \tfrac{1}{2}(w - y)^2 + 2\tilde{\lambda}, \tag{28}$$

where $w_x = \tilde{u}_1$ and $y_x = \tilde{u}_2$.

This can be derived from the (generalized) Miura map $\tilde{u} = v^2 + v' + \tilde{\lambda}$ (where $\tilde{\lambda}$ is a constant), which relates the KdV and mKdV hierarchies, by noting that the mKdV flow hierarchy (see equation (17)) is invariant under $v \rightarrow -v$. So we can have two solutions of KdV, $\tilde{u}_1$ and $\tilde{u}_2$, arising from the same solution, v, of the mKdV hierarchy.

We have $\tilde{u}_1 = w_x = v^2 + v' + \tilde{\lambda}$ and $\tilde{u}_2 = y_x = v^2 - v' + \tilde{\lambda}$, which give upon addition and subtraction:

$$w_x + y_x = 2v^2 + 2\tilde{\lambda}, \qquad w_x - y_x = 2v'. \tag{29}$$

The latter can then be integrated once and substituted into the former to give equation (28). This can then be specialized to solutions of the string equation by writing $\tilde{u}_i = t^{2\beta_k} u_i(xt^{\beta_k}) \equiv t^{2\beta_k} f_i'(xt^{\beta_k})$, which gives

$$w = t^{\beta_k} f_1(xt^{\beta_k}), \qquad y = t^{\beta_k} f_2(xt^{\beta_k}), \tag{30}$$

where a prime denotes differentiation with respect to $z = xt^{\beta_k}$. We have, therefore,

$$f_1' + f_2' = \tfrac{1}{2}(f_1 - f_2)^2 + 2\tilde{\lambda} t^{-2\beta_k}. \tag{31}$$

So we see that for consistency we must set $\tilde{\lambda} = 0$, which has an interesting interpretation to be discussed below. Hence we have

$$f_1' + f_2' = \tfrac{1}{2}(f_1 - f_2)^2. \tag{32}$$

So far, it is not clear that this transformation changes Γ. In order to establish the connection between this transformation and that displayed in equation (24), in which Γ appears explicitly, we first work perturbatively. Starting with the asymptotic expansions (6), for (say) u_1, it is easy to show that using equation (32) the asymptotic expansion obtained for u_2 is of the same form, except that Γ has been replaced by $\Gamma \pm 1$. We then expect that they are equivalent non-perturbatively, and have checked that this is the case by working numerically on some explicit solutions.

So we have established that the Bäcklund transformation takes us between solutions with asymptotics given in equation (6) which are for Γs differing from each other by unity. It is known [18] that the Bäcklund transformations (28) of KdV increase or decrease the soliton number of a given solution by unity. In fact, for a given solution u, the soliton corresponds to a bound state of $\mathcal{H}$ with (negative) eigenvalue $\tilde{\lambda}$, and the speed of the soliton is given by $-\tilde{\lambda}$ (up to a positive numerical constant). The soliton number of a particular solution can be read off by the number of such bound states and their individual velocities are set by the discrete spectrum.

In our case, we have established (so far) that there are no bound states in our solutions, and indeed, our restriction of the Bäcklund transformation forces $\tilde{\lambda} = 0$. So we deduce from this that Γ, which counts soliton number, must count the number of *zero-velocity* solitons. This is another sign that Γ is naturally an integer, and positive[11]. We will shortly find further evidence for this.

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[10] The $[\widetilde{P}, Q] = Q$ formalism of [6] allows any arbitrary combination of the t_ks to be switched on at once.

[11] It is amusing to note that one can generate non-trivial solutions of the string equation by starting with the trivial solution $u = 0$. This does not give solutions with the 'physical' asymptotics given in equation (6), which is not a contradiction, since we did not start with such a solution. What one obtains are pole solutions with the half-integer values of Γ similar to those discussed in [1, 15].
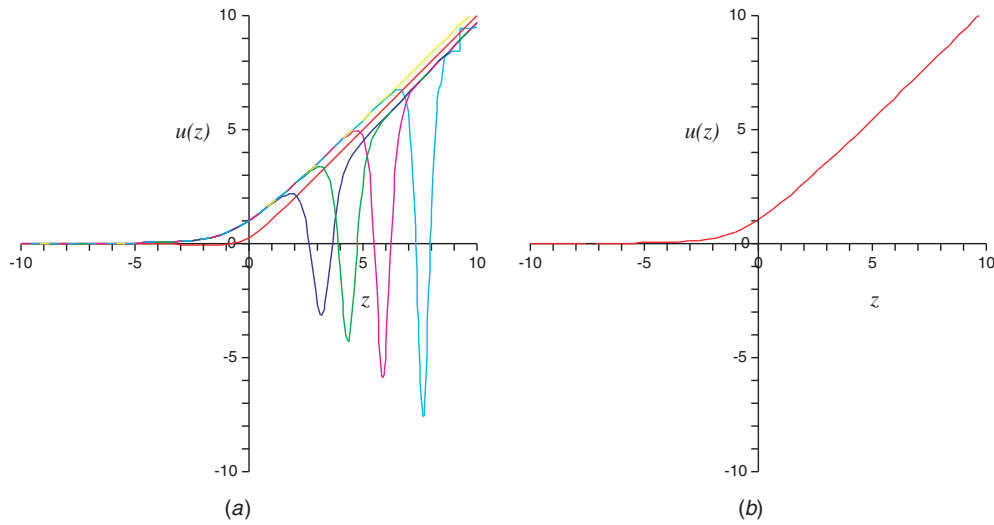
Figure 3. Numerical solutions to equation (1) for $u(z)$ in the case $k = 1$. The set of curves on the left shows $u(z)$ for a range of negative values of the parameter Γ, closely approaching the value -1 from above. The curve with the lowest well (very close to $\Gamma = -1$) is beginning to show signs of numerical error to the right. The curve on the right is the case of $\Gamma = +1$, used for comparison in the text.

Now that we have a method for generating new solutions $u(z)$ starting from old ones, we can return to our numerical study armed with more powerful tools.

4. The case of $\Gamma = -1$

We established in the last section that given a solution u_Γ, we can generate a solution $u_{\Gamma \pm 1}$. In the section before, we reported that it was difficult to solve the string equation numerically as Γ became more negative. Numerical precision was lost rapidly as Γ went below the value -0.9, while positive Γ is very much under control. We can use our transformation from the previous section to surmount this obstacle, by simply solving the equation numerically for $u(z)$ for $\Gamma = \epsilon$, for ϵ small and positive, and then use our transformation (24) to build a solution for $u(z)$ with $\Gamma = -1 + \epsilon$. We can simply follow ϵ to vanishingly small values and, therefore, learn the properties of $u(z)$ on the approach to $\Gamma = -1$.

We carried this out with very interesting results. See figure 3 for examples. We were able to use this method to generate $u(z)$ for $\Gamma = -1 + \epsilon$ where ϵ could easily be taken as small as $\sim 10^{-10}$. The potential well is observed to get deeper and narrow increasingly rapidly as $\epsilon \to 0$. In fact, we observe numerically that the well runs to infinity along the line $u(z) = -z$, becoming infinitely deep and narrow in the limit $\Gamma = -1$. Most interestingly, note that since the deepening of the well is accompanied by narrowing, the system has a chance of preventing the appearance of a dangerous bound state. We used the families of $u(z)$ functions found here as potentials for $\mathcal{H}$ and were indeed able to verify with some confidence (to within acceptable error tolerance) that there were no bound states present for any of the potentials.

So the case $\Gamma = -1$ is remarkably special, and in fact contains a surprise. At first, it seems that it develops a pathology, since as $\epsilon \to 0$ the well was seen to grow infinitely deep, but at the same time it became extremely narrow and, moreover, it moved off to infinity to the

right. So the solution may in fact be well behaved. In support of this, it is interesting to note the striking fact (again, from studying families of curves such as those displayed in figure 3) that in the limit the $\Gamma = -1$ curve looks rather like the $\Gamma = +1$ curve! The feature associated with the well is confined to a highly localized narrow (and deep) region, which has moved away. Everything that remains of the $\Gamma = -1$ curve at finite z falls (with a considerable numerical accuracy) on the $\Gamma = +1$ curve.

This numerical observation suggests the surprising result that the functions $u_{\Gamma=-1}$ and $u_{\Gamma=+1}$ are actually identical. The relationship is suggested by the limit, and an analytical proof is required to establish the result firmly. In fact, we can prove this for all k. The transformation (24) can be used to generate either of these functions starting with the case of $u_{\Gamma=0}$. Putting this function and $\Gamma = 0$ into the equation immediately gives the result that $u_{\Gamma=-1} = u_{\Gamma=+1}$, since there is no explicit appearance of Γ in the transformation in order to generate the difference between decreasing versus increasing Γ. From here it is easy to deduce the result that for Γ integer, $u_{-\Gamma} = u_{\Gamma}$.

This situation is familiar from studies of various special functions. An example is the Bessel functions, $J_n(x)$. When n is the integer it is smooth at the origin, and $J_{-n}(x) = (-1)^n J_n(x)$. When n is not an integer, however, $J_{-n}(x)$ and $J_n(x)$ are independent functions, and $J_{-n}(x)$ (for n positive) has a pole at the origin. In fact, the function $-J_{-1+\epsilon}(x)$ (ϵ positive) resembles $J_1(x)$ increasingly accurately as $\epsilon \to 0$, except for a small region near the origin where it diverges. This limiting situation is exactly what we have in the present case[12].

Note that below $\Gamma = -1$, there is a transition to a completely new behaviour. There, the system no longer has a smooth solution with the asymptotics given in equation (6), and there must be a singularity at the *finite z*. The origin of the singularity can be seen (for $k = 1$) by considering the form of the transformation (24). The denominator of the transformation is $4\mathcal{R}^2$, where, for the large positive z, we have that

$$\mathcal{R} = u^k - z \simeq \frac{\nu\Gamma}{z^{\frac{1}{2k}}} + \cdots, \tag{33}$$

and so for $\Gamma < 0$, the $k = 1$ solution must approach the line $u = z$ from below as $z \to +\infty$. Since $u(z)$ started out above this line (at $z \to -\infty$), it must be that it crosses the line somewhere in the interior. If this is the case, then somewhere in the interior, the new solution $u_{\Gamma-1}$ must have a singularity, since the denominator vanishes there. We expect similar arguments to hold for the negative Γ behaviour of other values of k.

So we have the very natural physical situation singling out Γ as a positive integer, since any solution $u(z)$ with $-1 < \Gamma < 0$ can be used as a seed for our transformation (24) to generate a $u(z)$ solution with poles, which have a problematic interpretation. Furthermore, any solution $u(z)$ for a positive fractional value of Γ can also be used, by applying the transformation successively, to generate such solutions, and so they are in the same class as the $-1 < \Gamma < 0$ solutions. The $u(z)$ for positive integer values of Γ are, therefore, special, in that they are not connected by our transformation to any solutions with poles.

5. Summary

So we see that studying the string equation non-perturbatively using a combination of exact and numerical methods reveals that the solutions have non-perturbative sensitivity to the values of Γ. In particular, it appears that for $\Gamma \geqslant -1$, the spectrum of the model is continuous and

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[12] We thank Johannes Walcher for pointing out this analogy.

bounded from below by zero, which is consistent with perturbative expectations. This means, for example, that the wavefunction of the extended brane [24] probe [25], the wavefunction ψ of $\mathcal{H}$, thought of as a function $\psi(\lambda_s)$ has a connected support.

For $\Gamma < -1$, the system does not seem to maintain a continuous, bounded spectrum. $\Gamma = -1$ is a special value, therefore, marking the beginning of a new phase of the model. It is tempting to think of this as a phase transition, but since we have every reason to think of Γ as a discrete parameter, this is probably not well motivated.

We are able to reach all positive integer values of Γ using our special Bäcklund transformation, and we observe that the cases of Γ being a negative integer do not really exist, since $\Gamma = -1$ is the same as $\Gamma = +1$, so successive application of the transformation will not generate any new negative integer cases. In other words, $u_{-\Gamma} = u_\Gamma$ when Γ is an integer. So the positive integer Γ solutions constitute a very special set. Meanwhile, fractional values of Γ can be connected by the transformation to solutions with poles, and so seem less physical. So while we have not rigorously proven that the positive integers are the only physical values selected by the string equation, it is certainly encouraging to find that the equation has properties that single out the positive integers as special.

This connects rather nicely to the rectangular matrix model understanding of the role of Γ mentioned in the first section: if M is 'row-like' ($\Gamma > 0$), then the combination $MM^\dagger$ will have $N + \Gamma$ eigenvalues, and precisely Γ of them are zero. These Γ extra zeros correspond to the zero-velocity KdV solitons (see below). On the other hand, if the matrix M is a 'column-like' ($\Gamma < 0$) rectangular matrix (i.e., more columns than rows) then the combination $MM^\dagger$ will have $N - |\Gamma|$ eigenvalues. So negative Γ has no extra zero eigenvalues, and does not correspond to having any background branes. In fact it seems to reduce the number of eigenvalues of the matrix, but at the large N (the limit in which we have defined the continuun model) this makes no difference. It would be interesting to determine what the negative Γ regime may mean physically[13]. This regime presumably has a plethora of bound states, which would need to be given a physical interpretation. Perhaps it is simply not possible to do so, and so the development of singularities in $u(z)$ for fractional Γ would be indicative of a non-perturbative sickness, reminiscent of that of the even k bosonic models [10–12].

The observation that there are Γ extra zero eigenvalues for the positive integer Γ fits very well with the fact that our specialization of the Bäcklund transformations creates and destroys KdV solitons with spectral parameter $\tilde{\lambda} = 0$. It is clear that the localized D-branes of the minimal string theory should be identified with these solitons. This direct connection of D-branes with the solitons of the underlying integrable system is intriguing and may well lead to interesting new physics in both string theory and in the theory of integrable systems. It deserves further investigation.

Acknowledgments

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[13] An attempt at interpreting negative integer Γ as corresponding to adding Γ anti-branes cannot be correct (we thank Nathan Seiberg for a comment about this), particularly in view of our observation about the non-existence of a distinct branch of solutions for $u(z)$ in this regime.

Note added in proof. While this paper was being written, a paper appeared [20] which contains results that have some overlap with ours.

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